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CMBR constraints on R^2 gravity

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Considering the inflation model based on a $f(R)$ gravity theory, we obtain several important constraints from the large angular scale CMBR observations. First, the ordinary slow-roll assumption during the inflation together with Harrison-Zel'dovich spectral conditions chooses R^2 gravity as a unique candidate. Second, the R^2 gravity leads to specific near scale-invariant Harrison-Zel'dovich spectra both for the scalar and the tensor perturbations. Third, using the COBE-DMR data we derive the strong constraints on the coupling constant and the energy scale during the inflation. Also, our result shows the gravitational wave contribution to the CMBR anisotropy is negligible. So, the future observation can provide the strong constraints on the the inflation model based on R^2 gravity. This is a summary of a talk presented in COSMO-01, and the more complete published version can be found in astro-ph/0102423.

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1 Introduction

The inflation stage in the early universe provides a natural mechanism which generates the origin of the large scale structures and the gravitational wave background. Thus, naturally, the observed large-scale structures and the gravitational wave background can constrain the model parameters of the proposed inflation models. In this context, the cosmic microwave background radiation (CMBR) temperature and polarization anisotropies in the large angular scale provide important constraints on the inflation models which are usually based on the scalar field or the generalized gravity theories. There have been many attempts to constrain the model parameters of simple models based on a single field potential in Einstein gravity. Recenlty, there have been growing interests on the roles of the generalized gravity theories other than the Einstein gravity in the early universe. These generalized gravity theories naturally arise either from attempts to quantize the gravity or as the low-energy limits of the unified theories including gravity. Especially, the R^2 terms appear in many theories like Kaluza-Klein models, string/M-theory programs, etc. The R^2 gravity has a natural reheating mechanism, so provides a self-contained inflation model without introducing a field or a phase transition [1].

We use the CMBR temperature anisotropy in the large angular scale, and derive constraints on the inflationary models based on the $f(R)$ gravity theory. We consider general $f(R)$ term in the Lagrangian and show that the Harrison-Zel'dovich spectrum [2] with the slow roll assumption during the inflation chooses the R^2 gravity as a unique candidate. Our derived constraints include the coupling constant of R^2 term and the energy scale during the inflation. We also show that the gravitational wave contribution to the CMBR anisotropy is suppressed.

2 Classical evolution

We consider a Lagrangian [3, 4]

$$L = \frac{1}{2} f(\phi, R) - \frac{1}{2}\omega(\phi)\phi^{;a}\phi_{,a} - V(\phi), \tag{1}$$

where $f(\phi, R)$ is a general algebraic function of the scalar field ϕ and the scalar curvature R; $\omega(\phi)$ and $V(\phi)$ are general algebraic functions of ϕ. We introduce $F \equiv \partial f/\partial R$.

We consider a spatially homogeneous and isotropic Friedmann world model together with the most general scalar- and tensor-type spacetime dependent perturbations [5]

$$\begin{aligned} ds^2 &= -a^2 (1 + 2\alpha)\, d\eta^2 - 2a^2 \beta_{,\alpha} d\eta dx^\alpha \\ &\quad + a^2 \left(\delta_{\alpha\beta} + 2\varphi\delta_{\alpha\beta} + 2\gamma_{,\alpha\beta} + 2C_{\alpha\beta}\right) dx^\alpha dx^\beta, \end{aligned} \tag{2}$$

where the transverse-tracefree $C_{\alpha\beta}$ indicates the gravitational wave. We also decompose $\phi = \bar{\phi} + \delta\phi$, $F = \bar{F} + \delta F$, etc.

The perturbed action in a unified form is given by [4]

$$\delta^2 S = \frac{1}{2}\int a^3 Q\left(\dot{\Phi}^2 - \frac{1}{a^2}\Phi^{|\gamma}\Phi_{,\gamma}\right) dt d^3 x. \tag{3}$$

This applies for the second order gravity such as either $f = f(\phi)R$ in the presence of ϕ or $f = f(R)$ without a field. For the scalar- and tensor-type perturbations [3, 4]

$$\Phi = \varphi_{\delta\phi}, \quad Q = \frac{\omega\dot{\phi}^2 + \frac{3\dot{F}^2}{2F}}{\left(H + \frac{\dot{F}}{2F}\right)^2}, \tag{4}$$

$$\Phi = C^\alpha_\beta, \quad Q = F, \tag{5}$$

where $H \equiv \dot{a}/a$, $\varphi_{\delta\phi} \equiv \varphi - H\delta\phi/\dot{\phi}$ is the gauge invariant variable.

The equation of motion and the large scale solution are given by [3, 4]

$$\frac{1}{a^3 Q}(a^3 Q\dot{\Phi})^{\cdot} - \frac{\Delta}{a^2}\Phi = 0, \tag{6}$$

$$\Phi = C(\mathbf{x}) - D(\mathbf{x})\int_0^t \frac{dt}{a^3 Q}. \tag{7}$$

Ignoring the transient solution we have a temporally conserved behavior

$$\Phi(\mathbf{x}, t) = C(\mathbf{x}). \tag{8}$$

Using $z \equiv a\sqrt{Q}$ and $v \equiv z\Phi$ eq. (6) becomes

$$v'' + \left(k^2 - z''/z\right) v = 0, \tag{9}$$

where a prime denotes the time derivative with respect to η.

3 Quantum generation

We introduce the slow-roll parameters [3]

$$\epsilon_1 \equiv \frac{\dot{H}}{H^2}, \quad \epsilon_2 \equiv \frac{\ddot{\phi}}{H\dot{\phi}}, \quad \epsilon_3 \equiv \frac{1}{2}\frac{\dot{F}}{HF}, \quad \epsilon_4 \equiv \frac{1}{2}\frac{\dot{E}}{HE}, \tag{10}$$

where $E \equiv F\left(\omega + \frac{3\dot{F}^2}{2\dot{\phi}^2 F}\right)$.

Assuming $\dot{\epsilon}_i = 0$ we have $z''/z = n/\eta^2$ where for the scalar- ($n = n_s$) and tensor ($n = n_t$) perturbations [3]

$$n_s = \frac{(1 - \epsilon_1 + \epsilon_2 - \epsilon_3 + \epsilon_4)(2 + \epsilon_2 - \epsilon_3 + \epsilon_4)}{(1 + \epsilon_1)^2},$$
$$n_t = \frac{(1 + \epsilon_3)(2 + \epsilon_1 + \epsilon_3)}{(1 + \epsilon_1)^2}. \tag{11}$$

For constant n, above equation becomes a Bessel's equation.

In the large-scale limit, the general power spectra based on vacuum expectation values lead to [4]

$$\mathcal{P}^{1/2}_{\hat{\varphi}_{\delta\phi}} = \frac{1}{\sqrt{Q}} \frac{H}{2\pi} \frac{1}{aH|\eta|} \frac{\Gamma(\nu_s)}{\Gamma(3/2)} \left(\frac{k|\eta|}{2} \right)^{3/2-\nu_s}, \quad (12)$$

$$\mathcal{P}^{1/2}_{\hat{C}_{\alpha\beta}} = \sqrt{\frac{2}{Q}} \frac{H}{2\pi} \frac{1}{aH|\eta|} \frac{\Gamma(\nu_t)}{\Gamma(3/2)} \left(\frac{k|\eta|}{2} \right)^{3/2-\nu_t}, \quad (13)$$

where $\nu_s \equiv \sqrt{n_s + 1/4}$ and $\nu_t \equiv \sqrt{n_t + 1/4}$, and we neglected the dependences on the vacuum choice.

4 Observations and the spectral constraints

In [3] it is shown that $\varphi_{\delta\phi}$ and $C_{\alpha\beta}$ are generally conserved independently of changing gravity theory, changing potential, and changing equation of state, as long as the scale remains in the large-scale; this applies to the case of the observationally relevant scales before the second horizon crossing in the matter dominated era. Using these conserved properties, we can identify the power spectra based on the vacuum expectation value during the inflation era [$\mathcal{P}_{\hat{\varphi}_{\delta\phi}}$ and $\mathcal{P}_{\hat{C}_{\alpha\beta}}$] with the classical power spectra based on the spatial average [$\mathcal{P}_{\varphi_{\delta\phi}}$ and $\mathcal{P}_{C_{\alpha\beta}}$]. Then, we have the same results in eqs. (12,13) for $\mathcal{P}_{\varphi_{\delta\phi}}$ and $\mathcal{P}_{C_{\alpha\beta}}$. Thus, eqs. (12,13) remain valid even in the matter dominated era. The spectral indices of the scalar- and tensor-type structures are given as $n_S - 1 = 3 - \sqrt{4n_s + 1}$ and $n_T = 3 - \sqrt{4n_t + 1}$. For the scale independent spectra the quadrupole anisotropy is [6]

$$\begin{aligned} \langle a_2^2 \rangle &= \langle a_2^2 \rangle_S + \langle a_2^2 \rangle_T \\ &= \frac{\pi}{75} \mathcal{P}_{\varphi_{\delta\phi}} + 7.74 \frac{1}{5} \frac{3}{32} \mathcal{P}_{C_{\alpha\beta}}. \end{aligned} \quad (14)$$

Then the ratio between the two perturbations becomes

$$r_2 \equiv \frac{\langle a_2^2 \rangle_T}{\langle a_2^2 \rangle_S} = 3.46 \frac{\mathcal{P}_{C_{\alpha\beta}}}{\mathcal{P}_{\varphi_{\delta\phi}}}. \quad (15)$$

The four-year *COBE*-DMR data provide [7]

$$\langle a_2^2 \rangle \simeq 1.1 \times 10^{-10}. \quad (16)$$

Now, we consider a special case with

$$L = \frac{1}{2} f(R). \quad (17)$$

The background equations are [3]

$$H^2 = \frac{1}{3F}\left(\frac{RF-f}{2} - 3H\dot{F}\right), \tag{18}$$

$$\dot{H} = -\frac{1}{2F}\left(\ddot{F} - H\dot{F}\right), \tag{19}$$

with $R = 6(2H^2 + \dot{H})$. Equation (19) gives

$$\epsilon_1 = \epsilon_3(1-\epsilon_4). \tag{20}$$

As the slow-roll conditions, we assume

$$|\epsilon_1| \ll 1. \tag{21}$$

We have $\epsilon_2 = 0$. The condition $\dot{\epsilon}_i = 0$ is reasonable in the sense that the large-scale structures are generated from a short duration (about 60 e-folds) of the inflation and the time variation of ϵ_i during that period is negligible. For constant ϵ_i we have $f \propto R^{(2+\epsilon_1)/(1+\epsilon_1-2\epsilon_3)}$. From eqs. (20,21) we have $\epsilon_3 \simeq 0$ or $\epsilon_4 \simeq 1$. In the case of $\epsilon_4 \simeq 1$, the spectral constraint $n_S - 1 \simeq 0$, thus $n_s \simeq 2$ in eq. (11) leads to $\epsilon_3 \simeq 1$ or 4. Both are excluded if we use $n_T \simeq 0$ thus $n_t \simeq 2$ in eq. (11). So, we have $\epsilon_3 \ll 1$ and

$$f \propto R^{2-\epsilon_1+4\epsilon_3}. \tag{22}$$

Using $\epsilon_3 \simeq 0$ in eq. (11) with $n_s \simeq 2$ gives $\epsilon_4 \simeq 0$ or -3; correspondingly we have $\epsilon_1 \simeq \epsilon_3$ or $4\epsilon_3$. For $\epsilon_4 \simeq -3$ we have exactly $f = R^2$ to the linear order in ϵ_i. If $\epsilon_4 \simeq 0$, then $\epsilon_3 = \epsilon_1$, thus

$$f \propto R^{2+3\epsilon_1}. \tag{23}$$

Our result shows that near scale-invariant Harrison-Zel'dovich spectra for *both* scalar- and tensor-type structures generated from inflation based on $f(R)$ gravity choose $f \propto R^2$ gravity. However, the current observations of CMBR provide a constraint on only the sum of the two types of structures. Usually, the observations provide constraint on the scalar-type perturbation, thus its amplitude and the spectral index. In such a case, our adoption of the Harrison-Zel'dovich type spectral index for the tensor-type structure as well to sort out the R^2 may be regarded as having a loop-hole in the argument. However, since we have used the n_T constraint to exclude the cases of $\epsilon_4 \simeq 1$, thus $\epsilon_3 \simeq 1$ or 4 just above eq. (22) we have $f \propto R^{-2}$ or $R^{-2/7}$, respectively. The corresponding values of the spectral index for the tensor-type perturbation are $n_T \simeq -2$ or -8, respectively, which are far from the expected values.

For $|\epsilon_i| \ll 1$ the spectral indices of the scalar- and the tensort-type perturbations become

$$n_S - 1 \simeq 2(3\epsilon_1 - \epsilon_4), \quad n_T \simeq 0, \tag{24}$$

to ϵ_i-order.

5 COBE-DMR constraints

We consider the following gravity during inflation

$$f = \frac{m_{pl}^2}{8\pi}\left(R + \frac{R^2}{6M^2}\right). \tag{25}$$

R^2 inflation model has been investigated extensively [1, 8]. During inflation we *assume* the second term dominates over the Einstein action which *requires* $H_{\rm infl}^2 \gg M^2$; this is a slow-roll regime and we have $|\epsilon_i| \ll 1$. For the model in eq. (23), the pure R^2 gravity implies $\epsilon_1 = 0$ and exponential inflation. However, in our case of eq. (25) we do have ϵ_1. In the slow-roll regime we have $\epsilon_3 = \epsilon_1$, thus eq. (24) remains valid. Assuming $H^2 \gg M^2$ we derive the power spectra

$$\mathcal{P}_{\varphi_{\delta F}}^{1/2} = \frac{1}{2\sqrt{3\pi}}\frac{1}{|\epsilon_1|}\frac{M}{m_{pl}}, \quad \mathcal{P}_{C_{\alpha\beta}}^{1/2} = \frac{1}{\sqrt{\pi}}\frac{M}{m_{pl}}, \tag{26}$$

where we ignored $[1+\mathcal{O}(\epsilon_i)]$ factors.

Then, we have

$$r_2 = 41.6\epsilon_1^2. \tag{27}$$

Comparing with the minimally coupled scalar field case where $r_2 = -6.93 n_T$ called consistency relation, in R^2 gravity r_2 in eq. (27) is quadratic order in ϵ_1, and n_T in eq. (24) vanishes to the linear order in ϵ_i. Therefore, in order to check whether the consistency relation remains in the R^2 gravity we need to consider the second-order effects of the slow-roll parameters.

Eqs. (14,26,16) give

$$\frac{M}{m_{pl}} = 3.1 \times 10^{-4}|\epsilon_1|. \tag{28}$$

Many works have shown the similar constraint [8]. While the previous works are based on an asymptotic solution of the background R^2-gravity inflation model, ours are based on exact solutions available when $\dot{\epsilon}_i = 0$. We can handle the perturbations analytically and derive the solution without using the conformal transformation.

From eqs. (18,19) we derive $\dot{H}^2 - 6H^2\dot{H} - H^2M^2 - 2H\ddot{H} = 0$. In the limit $H^2 \gg M^2$, and with the neglible $\ddot{H}$ term, we have a solution during the inflation as $H = H_i - M^2(t - t_i)/6$. Assuming that the inflation ends near $(t_e - t_i) \sim 6H_i/M^2$, we can estimate the number of e-folds from t_k (when the relevant scale k exits the Hubble horizon and reaches the large-scale limit) till the end of inflation as

$$N_k \equiv \int_{t_k}^{t_e} H dt = -\frac{H^2}{2\dot{H}}(t_k) = -\frac{1}{2\epsilon_1(t_k)}. \tag{29}$$

Thus $H(t_k)/M = \sqrt{N_k/3}$. Since the large-angular CMBR scales exit the horizon about $50 \sim 60$ e-folds before the end of the latest inflation, $N_k \sim 60$ gives $|\epsilon_1| = \frac{1}{2N_k} \sim 0.0083$. Then, we derive

$$r_2 \sim 0.0029, \quad \frac{M}{m_{pl}} \sim 2.6 \times 10^{-6}, \quad \frac{H(t_k)}{M} \sim 4.5. \tag{30}$$

The result shows the negligible gravitational wave contribution to CMBR anisotropies. Also, we can show that $\dot{\epsilon}_1 = -2\epsilon_1^2$ and $\ddot{H} \sim \mathcal{O}(\epsilon_1^3)$, thus $\epsilon_4 = \epsilon_1$ and eq. (24) becomes $n_S - 1 \simeq 4\epsilon_1 \sim -2/N_k$, thus

$$n_S - 1 \simeq -0.033, \quad n_T \simeq 0.00. \tag{31}$$

6 Conclusions

We show that R^2 gravity gives nearly scale-invariant Harrison-Zel'dovich spectra considering the slow-roll parameters to the linear order, i.e., $\mathcal{O}(\epsilon_i)$. The recent measurements of the CMBR by Boomerang and Maxima-1 in small angular scale [9] together with *COBE*-DMR data [10] provide the scalar spectral index $n_S - 1 \simeq 0.01^{+0.17}_{-0.14}$ at the 95% confidence level [11]. Thus, the inflation model considered in this work can be a generation mechanism for the large-scale structures. Also, for a successful inflation the gravitational wave contribution to the CMBR anisotropies should be negligible compared with the one of the scalar type perturbation. We expect that future observations of the spectral indices and the contribution of the gravitational wave to CMBR temperature/polarization anisotropies will test the considered inflation model. The original version of this proceeding was published in [12].

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